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on the left-hand side of this equation is $G^2a^6(27J^2 - I^3) = G^2a^6\Delta$, where Δ is the discriminant of the biquadratic under discussion. This proves the theorem proposed. It is of course understood that G^2 is *different* from zero.

To continue the discussion further under the preceding Question, write the right-hand side of the above expression in the form

$$F_1(x)[C_0x^2 + C_1x + C_2] + F(x)\Phi(x).$$

It may be shown that

$$\begin{aligned}
C_0 &= 9HG^2 - 9\lambda H^2 + \lambda^2 = 9HG^2 + (3H^2 - a^2I)\lambda = (a^2I)^2 - 6H^2a^2I - 9Ha^3J,\\
C_1 &= 3[3a^3Ja^2I + 2G^2a^2I - H(a^2I)^2 - 18H^2a^3J]\\
&= 3[6H^2G^2 + \lambda(G^2 + 2H\lambda - 18H^3)]\\
&= 3[a^3Ja^2I + H(a^2I)^2 - 8H^3a^2I - 18H^2a^3J],\\
C_2 &= 9(G^4 - \lambda^2H^2) + \lambda^3 + 12G^2\lambda H - 18HG^2a^2I\\
&= a^6\Delta + 6G^2(3a^3J + Ha^2I).
\end{aligned}$$

Writing $x = 2e_1$ in the expression above and noting that $F(2e_1) = 0$, it is seen that

$$\frac{1}{8e_1^3 + 12He_1^2 + G^2} = \frac{4C_0e_1^2 + 2C_1e_1 + C_2}{G^2a^6\Delta}.$$

Returning to the integral relations in the preceding Question it is seen that

$$\begin{aligned}
G^2a^6\Delta\tau &= B_0\sigma^3 + B_1\sigma^2 + B_2\sigma + B_3,\\
G^2a^6\Delta\sigma &= B_0\tau^3 + B_1\tau^2 + B_2\tau + B_3,
\end{aligned}$$

where the B's are integral functions of the second degree in e_1. The coefficients of these latter functions are integral in H, G^2 and I. Writing for σ, τ the values z_0, z_1, z_2, z_3 and interchanging e_1 with e_2 and e_3, we have here six pairs of integral relations corresponding to the rational relations indicated in 5.

The last part of 5 follows from the first part by use of 4.

NOTE. To Question 36, proposed by Professor Hancock, his own reply, printed above, is the only one as yet received. A reply to part 1 of the question will be welcome.—EDITOR.

DISCUSSIONS.

We present this month two discussions, both related more closely to secondary than to collegiate mathematics, but of considerable interest also to teachers of the latter. Professor Johnson, in advocating, with much reason, the early use of the complex number and its representation by a point of the plane, has made several comments on which there may well be differences of opinion. It is hoped that expressions on such questions will be forthcoming from readers.

Professor McClenon states without argument certain modes of approaching the teaching of logarithms. Many, perhaps most, teachers will be able to express a preference regarding these diverse methods, without hesitation.

I. THE COMPLEX QUANTITY IN ELEMENTARY ALGEBRA.

BY W. WOOLSEY JOHNSON, U. S. Naval Academy.

It is the purpose of this note to advocate an earlier introduction, to the student of elementary algebra, of the geometrical interpretation of the complex quantity $a + bi$ (Argand's diagram) than seems usually to be made.

Immediately after the solution of quadratic equations, let it be pointed out that the case of so-called impossible or imaginary roots is but a fresh instance of the occurrence of apparent "anomalies" in the results of mathematical operations, of which several are already familiar to the student. These anomalies always deny the existence of an answer to certain questions in the field of quantity to which they apply. Thus the subtraction indicated by 7–10 denotes the impossibility of an answer in certain number-systems; but in others, notably in connection with the location of a point on a line (a zero point and a unit of length being assumed), such a result presents no difficulty, but gives rise to the so-called negative quantities, of which the very name embodies the earlier view *denying* their existence.

So too we speak of a number (that is, an *integer*) as one that can or cannot be divided by a given smaller number. But when dealing with a straight line all lengths are divisible, and accordingly our number system is enlarged to include fractions.

Again, the indirect operation of extracting a root gave rise to an extension of the field of number, and the name *surd* attached to such a result as $\sqrt{2}$, for instance, reminds us that the attempt to represent it in the field of rational quantity, at which we had hitherto arrived, resulted in a "reductio ad absurdum" proving its impossibility. Yet, by a geometrical construction, the square root of two can be determined with as much exactness as that of an "exact" square.

The new anomaly at which we are supposing the student to have now arrived merits, to be sure, a name implying the impossibility of an answer far more than does, for instance, the negative quantity, which, as a result, may indicate a loss where a gain was expected, or a position on the left where one on the right was assumed. The impossible or imaginary quantity of quadratic algebra on the other hand generally indicates the non-existence of that which was sought, as when we seek to divide a number into parts which shall have a given product. On generalizing the constants, we here find occasion for an interesting class of questions which concern limits of existence and non-existence, taking the form of problems of maxima and minima.

But the question remains, can we find a field of quantity in which questions resulting in quadratic equations can always find an answer?

Having denoted the roots of $x^2 + 1 = 0$ (the simplest quadratic which presents the anomaly) by $\pm i$, we find the roots of every quadratic to be expressible in one of the forms $a \pm b$ or $a \pm bi$, in which a and b, being quantities of the kind hitherto known, may be called "scalars," because measurable on a scale or straight line. But, whereas in the first case the parts may be merged into single scalars, they are in the second case *heterogeneous* and cannot be so merged. Space of two dimensions now presents us the opportunity to form another scale, just as it does when we seek to represent a function of a scalar associated with its independent variable. But whereas, in that case, we were free to choose the direction of the new scale, consideration of i as a turning factor fixes the direction of the scale of imaginaries. It is, in my opinion, not necessary to say that we have

invented the new numbers $a + bi$ and annexed them to complete our field of number; rather let us say that, as in the former case of the straight line for scalars, we have found, ready to hand, a field of quantity of the desired kind, namely, one[1] in which the roots of a quadratic are always possible.

The mode of performing geometrically the fundamental operations of algebra with complex quantities; the mode in which the idea of absolute magnitude is attached to them; the construction of the roots of unity with their introduction to the many-valued function; and later their application to the power series; can hardly fail to interest and stimulate the student.[2]

Finally we observe that, at this point, the student will have an intelligent appreciation of the meaning of the fact, which may now be stated (without proof, for which he is, of course, not yet prepared), that every equation of the nth degree has n roots; no new anomaly appearing in the solution of those of higher degree, although new incommensurables do occur; thus the system of complex quantities is a complete set reproducing itself through all the operations of algebra, direct and indirect.

II. On the Teaching of Logarithms.

By R. B. McClenon, Grinnell College.

How should the subject of logarithms be presented to the beginner? The usual method has been, to start with the laws of exponents, define logarithms as powers of the base, and thus infer the fundamental laws of operation. This approach, however, is extremely likely to prove confusing to the student, because it does not bring out clearly enough the functional relation involved; and this disadvantage is accentuated by the impossibility of giving a satisfactory discussion of the meaning of an irrational exponent until a later stage of the mathematics course.

Two other methods have been proposed as a substitute for this one. One suggestion is, to use the historical approach, as employed by Napier in his invention of logarithms.[3] This would begin with the fundamental idea of the relation between an arithmetic and a geometric progression, the notion of "base" being at first entirely ignored. With the help of graphical methods, and an abundance of concrete problems chosen from physics, economics, and finance, it is said to be possible to bring out clearly and vividly the relation between the exponential and logarithmic functions. The other suggestion is, to abandon frankly all attempt to *explain* logarithms at the outset, but rather to focus the attention upon the actual use of logarithmic tables, until the mechanical rules for computa-

[1] A former student of Kronecker once told me that Kronecker would never admit the existence of a complex quantity. He said "there was not any complex variable $x + iy$. There were two variables, x and y." Whatever we say to this verbal contention, we shall continue to say that it takes two quantities (*i. e.*, scalars) to determine the position of a point in a given plane.

[2] In this connection the writer may refer to a construction of the roots of the quadratic with complex coefficients given by him to the New York Mathematical Society in 1893 and, in abstract, in the *Bulletin*, Vol. 2, p. 171.

[3] See, for example, Cajori, "History of the Exponential and Logarithmic Concepts," Am. Math. Monthly, Vol. 20 (1913), pp. 5–14.